INTERMEDIATE JACOBIANS AND HODGE STRUCTURES OF MODULI SPACES

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Note: It was brought to our attention, by H. Esnault, that the hyperplane H in our thm 6.1.1 needs to be general, otherwise it's false. The counterexample is to take U to be the smooth part of a cone over an elliptic curve, and H to be a hyperplane passing through the vertex. The flaw stems from the equation $j_! \mathbb{R}i'_* = \mathbb{R}i_* j'_!$ in the penultimate paragraph of its proof. As pointed out to us by M. Bondarko, this equation can be justified for general H by an easy modification of the proof of condition b_H on pp 35-36 of A. Beilinson, "How to glue perverse sheaves", Springer Lect Notes 128, (1987). Note that the rest of the paper is unaffected by this change, since the theorem had only been applied in the generic case.

1. Introduction

We work throughout over the complex numbers $\mathbb{C}$, i.e. all schemes are over $\mathbb{C}$ and all maps of schemes are maps of $\mathbb{C}$-schemes. A curve, unless otherwise stated, is a smooth complete curve. Points mean geometric points. We will, as is usual in such situations, toggle between the algebraic and analytic categories without warning.

For a curve X, $\mathcal{SU}_X(n, L)$ will denote the moduli space of *semi-stable* vector bundles of rank n and determinant L. The smooth open subvariety defining the *stable locus* will be denoted $\mathcal{SU}^s_X(n, L)$. We assume familiarity with the basic facts about such a moduli space as laid out, for example in [22, pp. 51–52, VI.A] (see also Theorems 10, 17 and 18 of *loc.cit.*).

When L is a line bundle of degree coprime to n, the moduli spaces $\mathcal{SU}_X(n,L)$ and $\mathcal{SU}^s_X(n,L)$ coincide, and are therefore smooth and projective. The cohomology groups $H^i(\mathcal{SU}_X(n,L),\mathbb{Q})$ carry pure Hodge structures which can, in principle, be determined by using a natural set of generators (Atiyah-Bott [2]) and relations (Jeffrey-Kirwan [13]) for the cohomology ring; we will say more about this later. When the degree of L is not coprime to n and $g > 2$, the situation is complicated by the fact that $\mathcal{SU}_X(n,L)$ is singular and $\mathcal{SU}^s_X(n,L)$ nonprojective. Thus the cohomology groups of these spaces carry (a-priori) mixed Hodge structures, and it are these structures that we wish to understand. Our main results concerns the situation in low degrees.

Theorem 1.0.1. *Let $\imath(n,g) = 2(n-1)g-(n-1)(n^2+3n+1)-7$. Let X be a curve of genus $g \geq 2$. If $n \geq 4$ and $i < \imath(n,g)$ are integers, then for any pair of line bundles L, L' (not necessarily with the same degree) on X, the mixed Hodge structures $H^i(\mathcal{SU}^s_X(n,L),\mathbb{Q})$ and $H^i(\mathcal{SU}^s_X(n,L'),\mathbb{Q})$ are* (noncanonically) *isomorphic and are both pure of weight i.*

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This statement is a bit disingenuous, it is vacuous unless $g \geq 16$. The explicit determination of these Hodge structures is rather delicate. However general considerations show that these Hodge structures are semisimple and it is not difficult to write down all the potential candidates for the simple summands.

Corollary 1.0.1. *With the notation as above, for $i < \imath(n,\, g)$ any simple summand of*

$$H^i(\mathcal{SU}^s_X(n, L), \mathbb{Q})$$

is, up to Tate twisting, a direct summand of a tensor power of $H^1(X)$

For third cohomology, a more refined analysis yields:

Theorem 1.0.2. *Let X be a curve of genus $g \geq 2$, $n \geq 2$ an integer and L a line bundle on X. Let $\mathcal{S}^s = \mathcal{SU}^s_X(n,\, L)$.*

(a) *If $g > \frac{3}{n-1} + \frac{n^2+3n+3}{2}$ and $n \geq 4$, then $H^3(\mathcal{S}^s, \mathbb{Z})$ is a pure Hodge structure of type $\{(1,2),\, (2,1)\}$, and it carries a natural polarization making the intermediate Jacobian*

$$J^2(\mathcal{S}^s) = \frac{H^3(\mathcal{S}^s,\, \mathbb{C})}{F^2 + H^3(\mathcal{S}^s,\, \mathbb{Z})}$$

into a principally polarized abelian variety. There is an isomorphism of principally polarized abelian varieties $J(X) \simeq J^2(\mathcal{S}^s)$.

(b) *If $\deg L$ is a multiple of n, then the conclusions of* (a) *are true for $g \geq 3$, $n \geq 2$ except the case $g = 3, n = 2$.*

The word "natural" above has the following meaning: an isomorphism between any two $\mathcal{S}^s$'s as above will induce an isomorphism on third cohomology which will respect the indicated polarizations. As an immediate corollary, we obtain the following Torelli theorem:

Corollary 1.0.2. *Let X and X' be curves of genus $g \geq 3$, L and L' line bundles of (possibly different degrees) on X and X' respectively.*

(a) *Assume that $n \geq 4$ is an integer such that $g > \frac{3}{n-1} + \frac{n^2+3n+1}{2}$. If*

$$\mathcal{SU}^s_X(n,\, L) \simeq \mathcal{SU}^s_{X'}(n,\, L') \tag{1.1}$$

or if

$$\mathcal{SU}_X(n,\, L) \simeq \mathcal{SU}_{X'}(n,\, L') \tag{1.2}$$

then

$$X \simeq X'.$$

(b) *If $\deg L = \deg L'$ and the common value is a multiple of n, then the conclusions of* (a) *are true for $n \geq 2$, except the case $g = 3, n = 2$.*

Proof. Since $\mathcal{SU}^s_X(n,\, L)$ (resp. $\mathcal{SU}^s_{X'}(n,\, L')$) is the smooth locus of $\mathcal{SU}_X(n,\, L)$ (resp. $\mathcal{SU}_{X'}(n,\, L')$), therefore it is enough to assume (1.1) holds. By assumption $J^2(\mathcal{SU}^s_X(n,\, L)) \simeq J^2(\mathcal{SU}^s_{X'}(n,\, L'))$ as polarized abelian varieties. Therefore $J(X) \simeq J(X')$, and the corollary follows from the usual Torelli theorem. □

When $(n, \deg\, L) = 1$ (the "coprime case"), the second theorem (and its corollary with $\deg\, L = \deg\, L'$) has been proven by Narasimhan and Ramanan [18], Tyurin [24] (both in the range $n \geq 2$ and $g \geq 2$, except when $g = 2, n = 3$) and the special case of their results, when $n = 2$, by Mumford and Newstead [16]. In

the non-coprime case, Kouvidakis and Pantev [14] had proved a Torelli theorem for $\mathcal{SU}_X(n,\,L)$, i.e. the corollary under the assumption (1.2), with better bounds. In fact the full corollary can be deduced from this case. However the present line of reasoning is extremely natural, and is of a rather different character from that of Kouvidakis and Pantev. In particular, Theorem 1.0.2 will not follow from their techniques. In the special case where $n = 2$ and $L = \mathcal{O}_X$, Balaji [4] has shown a similar Torelli type theorem for Seshadri's canonical desingularization $N \to \mathcal{SU}_X(2, \mathcal{O}_X)$ (see [23]) in the range $g > 3$. [1]

Our strategy in the proof of both theorems is to use a Hecke correspondence to relate the cohomology of $\mathcal{SU}_X^s(n, L)$ with that of another moduli space $\mathcal{SU}_X(n, L'')$ where the the degree of L'' is coprime to n. When $n > 2$ the maps defining the Hecke correspondence are only rational. And this necessitates some rather long calculations to bound the codimensions of the indeterminacy loci. Once the basic geometric properties of the correspondence are established, the first theorem follows from some standard arguments in Hodge theory. For the second theorem, we need to make the isomorphism on third cohomology canonical, and to moreover impose an intrinsic polarization on the Hodge structure $H^3(\mathcal{SU}_X^s(n,\,L))$.

2. The Main Ideas

For the rest of the paper, we fix a curve X of genus g, $n \in \mathbb{N}$, $d \in \mathbb{Z}$ and a line bundle L of degree d on X. Let $\mathcal{S} = \mathcal{SU}_X(n,\,L)$ and $\mathcal{S}^s = \mathcal{SU}_X^s(n,\,L)$.

The main theorems will be proved in the final section of this paper. The broad strategy of our proofs are as follows:

Step 1.

Case 1. Assume, that d is not divisible by n. Since $\mathcal{SU}_X(n,\,L)$ is canonically isomorphic to $\mathcal{SU}_X(n,\,L \otimes \xi^n)$ for every line bundle ξ on X, we may assume that $0 < d < n$.

Fix a set $\chi = \{x^1, \dots,\, x^{d-1}\} \subset X$ of $d-1$ distinct points. Let

$$\mathcal{S}_1 = \mathcal{SU}_X(n,\,L \otimes \mathcal{O}_X(-D))$$

where D is the divisor $x^1 + \dots + x^{d-1}$.

Construct (in §3) a generalized Hecke correspondence consisting of a pair of rational maps

$$\mathcal{S}_1 \xleftarrow{\pi} \mathbb{P} \overset{\phi}{\dashrightarrow} \mathcal{S}^s \tag{2.1}$$

By construction, there will be an open subset $U \subset \mathbb{P}$ such that both $\pi|_U$ and $\phi|_U$ will be fiber bundles with fiber isomorphic to $(\mathbb{P}^{n-1})^{d-1}$. Estimates on the codimensions of the complements of U and its image in $\mathcal{S}^s$ will be given in §4. These estimates, together with some generalities on cohomology and Hodge theory to be established in §6, will imply that for small i, there are noncanonical isomorphisms of mixed Hodge structures

$$H^i(\mathcal{S}_1, \mathbb{Q}) \cong H^i(\mathcal{S}^s, \mathbb{Q}). \tag{2.2}$$

[1]Balaji states the result for $g \geq 3$, but his proof seems to work only for $g > 3$. (See Remark 6.1.1).

Therefore this reduces the proof of Theorem 1.0.1 to the case where L and L' have degree 1, and this will be treated in § 7. Moreover, for sufficiently large g, we have isomorphisms modulo torsion of (integral, pure) Hodge structures

$$H^1(X,\,\mathbb{Z})(-1) \xrightarrow{\sim} H^3(\mathcal{S}_1,\,\mathbb{Z}) \xrightarrow{\sim} H^3(\mathcal{S}^s,\,\mathbb{Z}), \tag{2.3}$$

where the first isomorphism is the slant product with a certain universal class (§7), and the second now depends canonically on X, L and χ (§8). "(-1)" above is the Tate twist. An isomorphism *modulo torsion* of integral pure Hodge structures means that the underlying map of the finitely generated abelian groups is an isomorphism on the free parts. In particular, if as above these Hodge structure have odd weights, the resulting map of the corresponding intermediate Jacobians is an isomorphism.

Case 2. If d is divisible by n then we may assume that $d = 0$. In this case, set $\chi = \{x\}$ for some point $x \in X$. Setting $D = x$, we construct

$$\mathcal{S}_1 = \mathcal{SU}_X(n,\, L \otimes \mathcal{O}_X(-D))$$

as before. In §5, we construct a Hecke correspondence analogous to the one above. However, now the map ϕ is regular and the codimension estimates are substantially better. This allows us to establish 2.3 with a much better bound on the genus.

Step 2. Assume that g is chosen sufficiently large. Our first task is to find a (possibly nonprincipal) polarization $\Theta(\mathcal{S}^s)$ on $J^2(\mathcal{S}^s)$ or equivalently on the Hodge structure $H^3(\mathcal{S}^s)$ which varies algebraically with X. The basic tools for constructing this are given in §6. Let

$$\psi_{X,L,\chi}\colon H^1(X)(-1) \xrightarrow{\sim} H^3(\mathcal{S}^s)$$

be the isomorphism given above, and

$$\phi_{X,L,\chi} : J(X) \to J^2(\mathcal{S}^s)$$

the corresponding isomorphism of abelian varieties. The isomorphisms vary algebraically with the data $(X,\, L,\, \chi)$. One can pull $\Theta(\mathcal{S}^s)$ back to get a second polarization on $J(X)$ which varies algebraically with $(X,\, L,\, \chi)$. If we can find a positive integer m (independent of $(X,\, L,\, \chi)$) so that $\Theta(\mathcal{S}^s) = m\Theta$ where Θ is the standard polarization, then it will follow that, after replacing $\Theta(\mathcal{S}^s)$ with $\frac{1}{m}\Theta(\mathcal{S}^s)$, that $\Theta(\mathcal{S}^s)$ is a principal polarization such that $(J(X), \Theta) \cong (J^2(\mathcal{S}^s), \Theta(\mathcal{S}^s))$ as required. Since everything varies well, we can assume that X is a sufficiently general curve in moduli. In this case, one checks that any polarization on $J(X)$ is a multiple of the Θ. The precise argument is given in §8.

3. The Biregular Hecke Correspondence

The results of this section will be used to treat the case where $d = \deg L$ is not divisible by n. As explained earlier, we may assume that $0 < d < n$. We will continue the notation from step 1 of the previous section. The degree of $L \otimes \mathcal{O}_X(-D)$ is 1, therefore $\mathcal{S}_1$ is smooth and there exists a Poincaré bundle $\mathcal{W}$ on $X \times \mathcal{S}_1$. Let $\mathcal{W}_1, \ldots, \mathcal{W}_{d-1}$ be the $d-1$ vector bundles on $\mathcal{S}_1$ obtained by restricting $\mathcal{W}$ to $\{x^1\} \times \mathcal{S}_1 = \mathcal{S}_1, \ldots, \{x^{d-1}\} \times \mathcal{S}_1 = \mathcal{S}_1$ respectively. Let $\mathbb{P}_k = \mathbb{P}(\mathcal{W}_k)$, $k = 1, \ldots, d-1$, where we use the convention $\mathbb{P}(W_i) = \mathbf{Proj}(S^*(W_i^*))$. Let $\mathbb{P}$ $(= \mathbb{P}_{X,L,\chi})$ be the product $\mathbb{P}_1 \times_{\mathcal{S}_1} \ldots \times_{\mathcal{S}_1} \mathbb{P}_{d-1}$.

3.1. **The map $\phi\colon \mathbb{P} \dashrightarrow \mathcal{S}$.** We need some notation:

- $\pi\colon \mathbb{P} \to \mathcal{S}_1$ is the natural projection;
- For $1 \le k \le d-1$, $\pi_k\colon \mathbb{P} \to \mathbb{P}_k$ is the natural projection;
- $\imath\colon Z \hookrightarrow X$ is the reduced subscheme defined by $\chi = \{x^1, \dots, x^{d-1}\}$.
- $\imath_k\colon Z_k \hookrightarrow X$, the reduced scheme defined by $\{x_k\}$, $k = 1, \dots, d-1$.
- For any scheme S,
  (i) $p_S\colon X \times S \to S$ and $q_S\colon X \times S \to X$ are the natural projections;
  (ii) $Z^S = q_S^{-1}(Z)$;
  (iii) $Z_k^S = q_S^{-1}(Z_k)$, $k = 1, \dots, d-1$. Note that Z_k^S can be identified canonically with S.

We will show — in 3.2 — that there is an exact sequence

$$0 \longrightarrow (1 \times \pi)^*\mathcal{W} \longrightarrow \mathcal{V} \longrightarrow \mathcal{T}_0 \longrightarrow 0 \tag{3.1}$$

on $X \times \mathbb{P}$, with $\mathcal{V}$ a vector bundle on $X \times \mathbb{P}$ and $\mathcal{T}_0$ (the direct image of) a line bundle *on the closed subscheme* $Z^{\mathbb{P}}$, which is universal in the following sense: If $\psi\colon S \to \mathcal{S}_1$ is a $\mathcal{S}_1$-scheme and we have an exact sequence

$$0 \longrightarrow (1 \times \psi)^*\mathcal{W} \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0 \tag{3.2}$$

on $X \times S$, with $\mathcal{E}$ a vector bundle on $X \times S$ and $\mathcal{T}$ a line bundle *on the closed subscheme* Z^S, then there is a unique map of $\mathcal{S}_1$-schemes

$$g\colon S \longrightarrow \mathbb{P}$$

such that,

$$(1 \times g)^*(3.1) \equiv (3.2).$$

The $\equiv$ sign above means that the two exact sequences are isomorphic, and the left most isomorphism $(1 \times g)^* \circ (1 \times \pi)^* \xrightarrow{\sim} (1 \times \psi)^*$ is the canonical one.

Let $U_1 \subset \mathbb{P}$ be the maximal open subset such that $\mathcal{V}|_{X \times \{t\}}$ is stable for each $t \in U_1$. We shall see that this is nonempty, thus the natural moduli map $U_1 \to \mathcal{S}^s$ determines a rational map $\phi : \mathbb{P} \dashrightarrow \mathcal{S}^s$. The geometric properties of the map $\phi|_{U_1}$ are not obvious, so in 3.3 we will shrink U_1 to an open subset U with more manageable properties.

3.2. **The universal exact sequence.** We begin by reminding the reader of some elementary facts from commutative algebra. If A is a ring (commutative, with 1), $t \in A$ a non-zero divisor, and M an A-module, then each element $m_0 \in M$ gives rise to an equivalence class of extensions

$$0 \longrightarrow M \longrightarrow E_{m_0} \longrightarrow A/tA \longrightarrow 0 \tag{3.3}$$

where $E_{m_0} = (A \bigoplus M)\,/A(t, m_0)$, and the arrows are the obvious ones. Moreover, if $m_0 - m_1 \in tM$, say

$$m_0 - m_1 = tm'$$

then the extension given by m_0 is equivalent to that given by m_1. In fact, one checks that

$$\begin{gathered} E_{m_0} \xrightarrow{\sim} E_{m_1} \\ (a, m) \mapsto (a, m - am') \end{gathered} \tag{3.4}$$

gives the desired equivalence of extensions. This is another way of expressing the well known fact that each element of $M/tM = \operatorname{Ext}^1(A/t, M)$ gives rise to an extension.

One globalizes to get the following: Let S be a scheme, $T \overset{\imath}{\hookrightarrow} S$ a closed immersion, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module, U an open neighbourhood of T in S, and $t \in \Gamma(U, \mathcal{O}_S)$ an element which defines $T \hookrightarrow U$, and which is a non-zero divisor for $\Gamma(V, \mathcal{O}_S)$ for any open $V \subset U$. Then every global section s of $\imath^*\mathcal{F} = \mathcal{F} \otimes \mathcal{O}_T$ gives rise to an equivalence class of extensions

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_T \longrightarrow 0. \tag{3.5}$$

Indeed, we are reduced immediately to the case $S = U$. We build up exact sequences (3.3) on each affine open subset $W \subset S$, by picking a lift $\tilde{s_W} \in \Gamma(W, \mathcal{F})$ of $s\,|\,W$. One patches together these exact sequences via (3.4).

Now consider $\mathbb{P} = \mathbb{P}_1 \times_{\mathcal{S}_1} \ldots \times_{\mathcal{S}_1} \mathbb{P}_{d-1}$. For each $k = 1, \ldots, d-1$, let $p_k\colon\ \mathbb{P}_k \to \mathcal{S}_1$ be the natural projection. We have a universal exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow p_k^*\mathcal{W}_k \longrightarrow B \longrightarrow 0$$

whence a global section $s_k \in \Gamma(\mathbb{P}_k, p_k^*\mathcal{W}_k(1))$. However, note that

$$p_k^*\mathcal{W}_k = (1 \times p_k)^*\mathcal{W}\,|\,Z_k^{\mathbb{P}_k}$$

where we are identifying $Z_k^{\mathbb{P}_k}$ with $\mathbb{P}_k$. By (3.5) we get exact sequences

$$0 \longrightarrow (1 \times \pi)^*\mathcal{W} \longrightarrow \mathcal{V}_k \longrightarrow \mathcal{O}_{Z_k^{\mathbb{P}}} \otimes L_k \longrightarrow 0$$

where L_k is the line bundle obtained by pulling up $\mathcal{O}_{\mathbb{P}_k}(-1)$. It is not hard to see that $\mathcal{V}_k$ is a family of vector bundles parameterized by $\mathbb{P}$. Gluing these sequences together — the k-th and the l-th agree outside $Z_k^{\mathbb{P}}$ and $Z_l^{\mathbb{P}}$ — we obtain (3.1).

Now suppose we have a $\mathcal{S}_1$-scheme $\psi\colon\ S \to \mathcal{S}_1$ and the exact sequence (3.2)

$$0 \longrightarrow (1 \times \psi)^*\mathcal{W} \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0$$

on $X \times S$. Restricting (3.2) to Z_k^S $(1 \le k \le d-1)$ one checks that the kernel of $(1 \times \psi)^*\mathcal{W}\,|\,Z_k^S \to \mathcal{E}\,|Z_k^S$ is a line bundle $\mathcal{L}_k$. Identifying Z_k^S with S, we see that $(1 \times \psi)^*\mathcal{W}\,|\,Z_k^S = \psi^*\mathcal{W}_k$. Thus $\mathcal{L}_k$ is a line sub-bundle of $\psi^*\mathcal{W}_k$. By the universal property of $\mathbb{P}_k$, we see that we have a unique map of $\mathcal{S}_1$-schemes

$$g_k\colon\ S \longrightarrow \mathbb{P}_k$$

such that $\mathcal{O}(-1)$ gets pulled back to $\mathcal{L}_k$. The various g_k give us a map

$$g\colon S \longrightarrow \mathbb{P}$$

One checks that g has the required universal property. The uniqueness of g follows from the uniqueness of each g_k.

3.3. **The biregular Hecke correspondence.** As explained earlier, we will need to consider an open subset $U \subset U_1$, with good geometric properties. Essentially U will be the preimage of a subset $U' \subset \mathcal{S}^s$ parameterizing those stable vector bundles E for which the kernel of $p : E \to \mathcal{O}_Z$ is stable for every surjection. We again remind the reader, that if $\deg L$ is a multiple of n, one can do better, as will be seen in section 5.

To construct U and U' rigorously, it would convenient to have some sort of universal family of vector bundles on $X \times \mathcal{S}^s$. Unfortunately, such a family will not exist when $(n, d) \neq 1$. To get around this, we can go back to the construction ([22]): $\mathcal{S}^s$ is obtained as a good quotient of an open subscheme R of a Quot scheme by a reductive group G (which is in fact a projective general linear group). There is a vector bundle $\mathbb{E}'$ on $X \times R$ whose restriction to $X \times \{r\}$ is the bundle represented

by the image of r in $\mathcal{S}^s$. For $x \in X$, let E_x be the restriction of $\mathbb{E}'$ to $R \cong \{x\} \times R$. There is a natural G action on $\mathbb{E}'$ which induces one on E_x for every $x \in X$ and hence on every $\mathbb{P}(E_x^*)$. This action is locally a product of the action on R with a trivial action along the fibers, therefore $\mathbb{P}(E_x^*)/G$ is a projective space bundle (or more accurately a Brauer-Severi scheme) over $\mathcal{S}^s$. Let $\pi' : \mathbb{P}' \to R$ be the fiber product of $\mathbb{P}(E_{x^i}^*) \to R$ for $i = 1, \dots d-1$. Then $\mathbb{P}'/G \to \mathcal{S}^s$ is a $(\mathbb{P}^{n-1})^{d-1}$-bundle i.e. a smooth morphism with fibers isomorphic to $(\mathbb{P}^{n-1})^{d-1}$. Let L_i be the pullback of $\mathcal{O}_{\mathbb{P}(E_{x^i}^*)}(1)$ to $\mathbb{P}'$. There is a canonical map

$$\kappa : (1 \times \pi')^* \mathbb{E}' \to \bigoplus_k i_{k*} L_k$$

where $i_k \colon \mathbb{P}' = \{x^k\} \times \mathbb{P}' \to X \times \mathbb{P}'$ is the natural closed immersion.

Let $\mathcal{U}''$ be the maximal open subset of $\mathbb{P}'$ such that $\ker(\kappa)|_{X \times \{t\}}$ is stable for each $t \in \mathcal{U}''$. Set $\mathcal{U}' = R \setminus \pi'(\mathbb{P}' \setminus \mathcal{U}'')$ and $\mathcal{U} = {\pi'}^{-1}\mathcal{U}' \subset \mathcal{U}''$. Both $\mathcal{U}$ and $\mathcal{U}'$ are invariant under G, and we let U and $U' \subset \mathcal{S}^s$ be the quotients. The map $U \to U'$ is again a $(\mathbb{P}^{n-1})^{d-1}$-bundle. The exact sequence

$$0 \to \ker(\kappa) \to (1 \times \pi')^* \mathbb{E}' \to \bigoplus_k i_{k*} L_k \to 0$$

yields, via the universal property of $\mathbb{P}$ a morphism $\mathcal{U} \to \mathbb{P}$ which factors through U. The map $U \to \mathbb{P}$ is an open immersion to a subset of U_1. Thus we get a diagram

$$\mathcal{S}_1 \leftarrow U \xrightarrow{f} \mathcal{S}^s \tag{3.6}$$

which will be referred to as the *biregular Hecke correspondence*.

Remark 3.3.1. With the above notation, note that $\mathbb{P}'/G \to \mathcal{S}^s$ is the fibre product of $\mathbb{P}(E_{x^i}^*)/G \to \mathcal{S}^s$ $(i = 1, \dots, d-1)$. It follows that f itself may be regarded as the fibre product of $\mathbb{P}^{n-1}$ bundles over U'.

4. Codimension Estimates

We will continue the notation from the previous section. Our goal is to establish the basic estimates on the codimensions of the complements of U and U'. In particular, these estimates will show that U and U' are nonempty.

4.1. **Special Subvarieties.** Let X be a smooth curve of genus $g \geq 2$. Fix integers $n > m \geq 1$, a rational number μ_0 and a line bundle M. Let $T_{m,\mu_0}(n, M)$ be the subset of $\mathcal{SU}^s(n, M)$ whose points correspond to bundles E for which there exists a subbundle $F \subset E$ of rank m and slope $\mu(F) = \mu_0$.

Lemma 4.1.1. *$T_{m,\mu_0}(n, M)$ is Zariski closed, and can therefore be regarded as a reduced subscheme. There exists a scheme Σ such that $X \times \Sigma$ carries a rank n vector bundle $\mathbb{E}$ with a rank m subbundle $\mathbb{F}$. For each $\sigma \in \Sigma$, the restriction of $\mathbb{E}$ to $X \times \{\sigma\}$ is stable of degree d, while the restriction of $\mathbb{F}$ has slope μ_0. The canonical map $\Sigma \to \mathcal{SU}^s(n, M)$ has T_{m,μ_0} as its image.*

Proof. The argument is similar to that used above. $S = \mathcal{SU}^s(n, M)$ can be realized as a good quotient of a subscheme R of a Quot scheme by a reductive group G. $X \times R$ carries a vector bundle $\mathbb{E}'$ whose restriction to $X \times \{r\}$ is the bundle represented by the image of r in S. $\mathbb{E}'$ extends to a coherent sheaf (denoted by the same symbol) on the closure $Y = X \times \bar{R}$. Let Q be the closed subscheme of the relative Quot

scheme $Quot_{X\times\bar{R}/\bar{R}}(\mathbb{E}')$ parameterizing quotients which restrict to vector bundles of degree $\deg M - \mu_0 m$ and rank $n-m$ on each $X\times\{r\}$. The the intersection $\tilde{S}$ of R with the image of the projection $Q\to\bar{R}$ is closed and G-invariant. Therefore the image of $\tilde{S}$ in S, which is T_{m,μ_0}, is closed.

Set $\Sigma = R\times_{\bar{R}} Q$ and $\mathbb{E}$ to the pullback of $\mathbb{E}'$ to $X\times\Sigma = (X\times R)\times_{\bar{R}} Q$. Then it is easily seen that these have the required properties. ☐

4.2. **Deformations.** Let $D = \operatorname{Spec}\mathbb{C}[\epsilon]/(\epsilon^2)$, and let $\mathcal{E}$ be a vector bundle on $X\times D$. X can be identified with a closed subscheme of $X\times D$ with ideal sheaf $\mathcal{I} = \epsilon O_{X\times D}$. Let $E = \mathcal{E}\otimes\mathcal{O}_X \cong \mathcal{E}\otimes\mathcal{I}$. Then there is an exact sequence

$$0\to E\to\mathcal{E}\to E\to 0$$

which is classified by an element of $Ext^1(E,E)\cong H^1(\mathcal{E}nd(E))$. Furthermore

$$0\to\det(E)\to\det(\mathcal{E})\to\det(E)\to 0$$

is classified by the trace of the above class. If $\mathcal{E}nd_0(E)$ denotes the traceless part of $\mathcal{E}nd(E)$, then $H^1(\mathcal{E}nd_0(E))$ classifies deformations of E which induce trivial deformations of $\det(E)$.

If $M = \det E$, then elements of $H^1(X,\mathcal{E}nd_0(E))$ give rise to maps from D to $\mathcal{SU}(n,M)$ which send the closed point to the class of E. This map is well known to yield an isomorphism between $H^1(X,\mathcal{E}nd_0(E))$ and the tangent space to $\mathcal{SU}(n,M)$ at $[E]$.

Let E be a vector bundle corresponding to general point $[E]$ of a component of T_{m,μ_0}, and let v be a tangent vector to T_{m,μ_0} based at $[E]$. This vector can be lifted to tangent vector $\tilde{v}$ to Σ at a point σ lying over $[E]$. Then $E\cong\mathbb{E}|_{X\times\{\sigma\}}$, and let F be the subbundle corresponding to $\mathbb{F}|_{X\times\{\sigma\}}$. Set $G = E/F$,

$$K = \ker[\mathcal{E}nd(E)\to\mathcal{H}om(F,G)]$$

and

$$K_0 = K\cap\mathcal{E}nd_0(E) = \ker[\mathcal{E}nd_0(E)\to\mathcal{H}om(F,G)].$$

Note that $K = K_0\oplus(id)\cdot\mathcal{O}_X$.

Let $\mathcal{E}$ be the first order deformation of E corresponding to the tangent vector v. Explicitly, if $D\to\Sigma$ is the map corresponding to $\tilde{v}$, then $\mathcal{E} = \mathbb{E}|_D$. Setting $\mathcal{F} = \mathbb{F}|_D$ gives a deformation of F which fits into a diagram:

$$\begin{array}{ccccccccc}
0 & \to & E & \to & \mathcal{E} & \to & E & \to & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & F & \to & \mathcal{F} & \to & F & \to & 0.
\end{array}$$

The images of the classes of $\mathcal{F}$ and $\mathcal{E}$ in $Ext^1(F,E)$ agree up to sign. Therefore the class of $\mathcal{E}$ lies in the kernel of the map to $Ext^1(F,G)$ which is the image of $H^1(X,K_0)$. Therefore:

Lemma 4.2.1. *The image of $H^1(X,K_0)$ in $H^1(X,\mathcal{E}nd_0(E))$ contains the tangent space to T_{m,μ_0} at E.*

A simple diagram chase shows that there is an exact sequence

$$0\to\mathcal{H}om(E,F)\to K\to\mathcal{E}nd(G)\to 0$$

Therefore

Corollary 4.2.1. $\dim T_{m,\mu_0}\le h^1(\mathcal{E}nd(F)) + h^1(\mathcal{H}om(G,F)) + h^1(\mathcal{E}nd(G)) - g.$

4.3. **Preliminary Codimension Estimates.** Let us say that T_{m,μ_0} is admissible if for any general point $[E]$ of a component of T_{m,μ_0} of largest dimension, there exists a semistable subbundle $F \subset E$ of rank m and slope μ_0. We will estimate codimension of an admissible $T_{m,\mu_0}(n, M)$ from below as a function of four quantities $g \geq 2$,$n \geq 2$, $n > m \geq 1$, and $\mu_0 < \mu = n/deg\, M$. The imposition of admissibility simplifies the calculations, and presents no real loss of generality. Fixing E and F as above, let $G = E/F$ and let $0 = G_0 \subset G_1 \subset \ldots \subset G_r = G$ be the Harder-Narasimhan filtration. Set $n_i = \mathrm{rk}(G_i/G_{i-1})$ and $n_0 = m$. We have

$$\mu < \mu\,(G_r/G_{r-1}) < \cdots < \mu(G_1) < \frac{n_0}{n_1}(\mu - \mu_0) + \mu$$

where the last inequality follows from the bound on the slope of the preimage of G_1 in E.

In the computations below, we will make use of the additivity of "deg", and "rk" and

Lemma 4.3.1. *If A and B are locally free then $\mu(A \otimes B) = \mu(A) + \mu(B)$ and $\mu(\mathcal{H}om(A, B)) = \mu(B) - \mu(A)$.*

Lemma 4.3.2. *If V is semi-stable, then $h^0(V) \leq \deg(V) + \mathrm{rk}(V)$ provided that the right side is nonnegative.*

Proof. This is obvious if $\deg V < 0$, since $h^0(V) = 0$. For $\deg V \geq 0$, we can assume that the lemma holds for $V(-p)$ by induction. Therefore

$$h^0(V) \leq h^0(V(-p)) + h^0(V/V(-p)) \leq (\deg V - r + r) + r.$$

□

Corollary 4.3.1. *If V is a semistable vector bundle then $h^1(\mathcal{E}nd(V)) \leq \mathrm{rk}(V)^2 g$.*

Proof. As $\mathcal{E}nd(V)$ is semistable, we have $h^0(\mathcal{E}nd(V)) \leq \mathrm{rk}(V)^2$, and the result now follows from Riemann-Roch. □

Heuristically, $\dim T_{m,\mu_0}$ should be given by the number of moduli for F, G_i plus extensions. To make this more rigorous, we will work infinitesimally, and appeal to corollary 4.2.1. Each of the terms of the corollary can be estimated in turn. For the first term, we have

$$h^1(\mathcal{E}nd(F)) \leq n_0^2 g \tag{4.1}$$

$Hom(G, F) = 0$ by the numerical conditions, therefore

$$h^1(\mathcal{H}om(G, F)) = -\chi(G^* \otimes F)$$

So Riemann-Roch yields

$$h^1(\mathcal{H}om(G, F)) = n_0(n_1 + \ldots n_r)(g - 1) + nn_0(\mu - \mu_0). \tag{4.2}$$

Note deg $\mathcal{H}om(G, F) =$ deg $\mathcal{H}om(E, F) = -nn_0(\mu - \mu_0)$.

The last term $h^1(\mathcal{E}nd(G))$ remains. If $G = G_1$ is semi-stable, then a bound is given as above. To analyze the general case, we need:

Lemma 4.3.3. *If V has a filtration such that the associated graded sheaves are semi-stable bundle with slope at least -1, then $h^1(V) \leq g \cdot \mathrm{rk}(V)$.*

Proof. By subadditivity of h^1, it is enough to assume that V is semi-stable with slope at least -1, in which case the result follows from the previous lemma together with Riemann-Roch. □

Corollary 4.3.2. *Let W be a semi-stable bundle, and V a vector bundle with a filtration such that the associated graded sheaves are semi-stable bundle with slope at least $\mu(W)-1$, then*

$$h^1(\mathcal{H}om(W,V)) \le \mathrm{rk}(V)\,\mathrm{rk}(W)g.$$

Lemma 4.3.4. *We have the inequality*

$$h^1(\mathcal{E}nd(G_k)) \le (n_1+\dots n_k)^2 g + (\sum_{1\le i<j\le k} n_i n_j)\,(\mu(G_1)-\mu(G)-1)\,.$$

Proof. From

$$0 \to G_{k-1} \to G_k \to G_k/G_{k-1} \to 0,$$

we obtain

$$\begin{aligned} h^1(\mathcal{E}nd(G_k)) \le & h^1(\mathcal{E}nd(G_{k-1})) + h^1(\mathcal{E}nd(G_k/G_{k-1})) \\ & + h^1(\mathcal{H}om(G_{k-1}, G_k/G_{k-1})) + h^1(\mathcal{H}om(G_k/G_{k-1}, G_{k-1})). \end{aligned}$$

It suffices to find upper bounds for each of the terms on the right, and then sum them. The first term can be controlled by induction, and the second and third terms by the previous corollaries. For the last term, an estimate can be obtained by combining $\mathrm{Hom}(G_k/G_{k-1}, G_{k-1}) = 0$, with

$$\begin{aligned} -\deg \mathcal{H}om(G_k/G_{k-1}, G_{k-1}) &= (n_1+\dots n_{k-1})n_k\,[\mu(G_k/G_{k-1})-\mu(G_{k-1})] \\ &= \sum_{i=1}^{k-1} n_i n_k\,[\mu(G_k/G_{k-1})-\mu(G_i/G_{i-1})] \\ &< \sum_{i=1}^{k-1} n_i n_k\,[\mu(G_1)-\mu] \end{aligned}$$

and the Riemann-Roch theorem. □

Thus

$$\begin{aligned} h^1(\mathcal{E}nd(G_r)) - g \quad \le \quad & \left[(n_1+\dots n_r)^2-1\right] g \\ & +(\sum_{1\le i<j\le r} n_i n_j)\,(\mu(G_1)-\mu-1) \end{aligned} \tag{4.3}$$

Subtracting equations (4.1), (4.2) and (4.3) from $\dim \mathcal{SU}(n,M)$, simplifying, and replacing n_0 by m, yields

$$\begin{aligned} \mathrm{codim}\, T_{m,\mu_0} > & m(n-m)(g+1) - (\sum_{1\le i<j\le r} n_i n_j)\,(\mu(G_1)-\mu(G)-1) \\ & - nm(\mu-\mu_0) - (n^2-1) \end{aligned}$$

To proceed further, we use $\sum_{1\le i<j\le r} n_i n_j \le \frac{1}{2}(n-m)^2$ and

$$\mu(G_1)-\mu \le \frac{m}{n_1}(\mu-\mu_0) \le m(\mu-\mu_0)$$

Putting these together yields

Proposition 4.3.1. *If T_{m,μ_0} is admissible, we have*

$$\operatorname{codim} T_{m,\mu_0} > m(n-m)(g+1) - \frac{1}{2}m(n-m)^2(\mu-\mu_0) - nm(\mu-\mu_0) - (n^2-1).$$

4.4. **Final Codimension Estimates.** Recall that in § 3 we had a diagram

$$\mathcal{S}_1 \xleftarrow{\pi} \mathbb{P} \supset U_1 \supset U \xrightarrow{f} U' \subset \mathcal{S}^s$$

where $\mathbb{P}$ could be viewed as the parameter space for extensions

$$0 \to E \to E' \to \mathcal{O}_Z \to 0$$

with $E \in \mathcal{S}_1$. An extension E' lies in $\mathbb{P} \setminus U_1$ if and only if there exists a subbundle $F' \subset E'$ with $\mu(F') \geq \mu(E') = \mu(E) + (d-1)/n$. F' can be assumed to be semistable, since otherwise it can be replaced by the first step in the Harder-Narasimhan filtration. Let $F = F' \cap E$, then

$$\mu(F) \geq \mu(F') - \frac{(d-1)}{m} \geq \mu(E) - (d-1)\left(\frac{1}{m} - \frac{1}{n}\right) \geq \mu(E) - \frac{n-1}{m}$$

where $m = \operatorname{rk}(F)$.

Similarly, if an extension E' lies in in $U_1 \setminus U$ then there exist a coherent subsheaf $F_1' \subset E'$ which violates stability of the kernel of some map $E' \to \mathcal{O}_Z$. After replacing F_1' by the maximal semistable subbundle of its saturation, and setting $F_1 = F_1' \cap E$, we obtain

$$\mu(F_1) \geq \mu(F_1') - \frac{(d-1)}{m'} \geq \mu(E) - \frac{(d-1)}{m'} \geq \mu(E) - \frac{n-1}{m'}$$

where $m' = \operatorname{rk}(F_1)$.

Therefore $\pi(\mathbb{P} \backslash U)$ is contained in the union of admissible $T_{m,\mu_0}(n, L \otimes \mathcal{O}_X(-D))$, as m varies between 1 and $n-1$, and $\mu - \mu_0$ between zero and $(n-1)/m$. Clearly $\operatorname{codim}(\mathbb{P} \backslash U) \geq \operatorname{codim}(\pi(\mathbb{P} \backslash U))$. Therefore a term by term estimate of the bound in Proposition 4.3.1, using $(n-m) \leq n-1$ and $m(n-m) \geq n-1$ (for $1 \leq m \leq n-1$), yields:

Theorem 4.4.1. $\operatorname{codim}(\mathbb{P} \setminus U) > (n-1)\left[g - \frac{1}{2}(n^2+n)\right]$

Corollary 4.4.1. *If*

$$g > \frac{k}{n-1} + \frac{n^2+n}{2}$$

then,

$$\operatorname{codim}(\mathbb{P} \setminus U) > k.$$

The estimate on the codimension of U' is obtained by a similar argument. If $E' \in \mathcal{S} \backslash U'$, then there exists a surjection $E' \to \mathcal{O}_Z$ and a subbundle $F \subset \ker[E' \to \mathcal{O}_Z]$ violates the stability of the kernel. Let $m = \operatorname{rk}(F)$ and F' be the maximal semistable sub-bundle of the saturation of F in E', then

$$\begin{aligned}
\mu(F) &\geq \mu(F') - \frac{(d-1)}{m} \\
&\geq \mu(E) - \frac{(d-1)}{m} \\
&= \mu(E') - (d-1)\left(\frac{1}{n} + \frac{1}{m}\right) \\
&\geq \mu(E') - (n-1)\left(\frac{1}{n} + \frac{1}{m}\right).
\end{aligned}$$

Thus the complement of U' lies in the union of admissible $T_{m,\mu_0}(n,\, L)$ as m varies between 1 and $n-1$, and $\mu-\mu_0$ varies between zero and $(n-1)(\frac{1}{n}+\frac{1}{m})$. An elementary analysis shows that

$$\left(1+\frac{m}{n}\right)\left[\frac{1}{2}(n-m)^2+n\right] \leq \left(1+\frac{1}{n}\right)\left[\frac{1}{2}(n-1)^2+n\right]$$

for $1 \leq m \leq n-1$ and $n \geq 4$. Combining this and the trivial estimate $m(n-m) \geq n-1$ with Proposition 4.3.1 yields:

Theorem 4.4.2. *If $n \geq 4$ then*

$$\operatorname{codim}(\mathcal{S}^s \setminus U) > (n-1)\left[g-\frac{n^2+3n+1}{2}-\frac{3}{n}\right] > (n-1)\left[g-\frac{n^2+3n+1}{2}\right]-3$$

Corollary 4.4.2. *If $n \geq 4$ and $g > \frac{k}{n-1}+\frac{n^2+3n+3}{2}$, then $\operatorname{codim}(\mathcal{S}^s \setminus U') > k$.*

Of course, the above inequality for g implies the inequality in corollary 4.4.1. The restriction on n above is harmless, because the cases of $n = 2,\, 3$ can be handled by the results of the next section.

5. Hecke When $\deg L = 0$

The notations in this section are special to this section. We now give a an improved Hecke correspondence when $\deg L \in n\mathbb{Z}$. Clearly, without loss of generality, we may (and will) assume that $\deg L = 0$. In this case, instead of choosing $d-1$ points on X (which clearly does not make sense), we choose one point $x \in X$, and let Z be the reduced scheme supported on $\{x\}$. Let D be the divisor given by $\{x\}$. Then $\deg L \otimes \mathcal{O}_X(-D)$ is -1, and if $\mathcal{S}_1 = \mathcal{SU}_X(n,\, L \otimes \mathcal{O}_X(-D))$, then $\mathcal{S}_1$ is smooth and there exists a Poincaré bundle $\mathcal{W}$ on $X \times \mathcal{S}_1$. Let $\mathcal{W}_1$ be the vector bundle on $\mathcal{S}_1$ obtained by restricting $\mathcal{W}$ to $\{x\} \times \mathcal{S}_1 = \mathcal{S}_1$, and $\mathbb{P} = \mathbb{P}(\mathcal{W}_1)$. Let $\pi : \mathbb{P} \to \mathcal{S}_1$ be the natural projection. Then as before we have the universal exact sequence (3.1)

$$0 \to (1 \times \pi)^*\mathcal{W} \to \mathcal{V} \to \mathcal{T}_0 \to 0,$$

of coherent sheaves on $X \times \mathbb{P}$ with $\mathcal{V}$ a vector bundle, and $\mathcal{T}_0$ a sheaf supported on $\{x\} \times \mathbb{P} = \mathbb{P}$ which is a line bundle on $\mathbb{P}$. This means that $\mathbb{P}$ parameterizes exact sequences

$$0 \to W \to V \to \mathcal{O}_Z \to 0$$

of coherent sheaves on X with $W \in \mathcal{S}_1$ and V a vector bundle (necessarily of rank n and determinant L).

There is a way of interpreting this universal property in terms of quasi-parabolic bundles (see [15, p. 211–212, Definition 1.5] for the definitions of quasi-parabolic and parabolic bundles). We introduce a quasi-parabolic datum on X by attaching the flag type $(1,\, n-1)$ to the point x. From now onwards *quasi-parabolic structures will be with respect to this datum and on vector bundles of rank n and determinant L.* One observes that for a vector bundle V (of rank n and determinant L), a surjective map $V \twoheadrightarrow \mathcal{O}_Z$ determines a unique quasi-parabolic structure, and two such surjections give the same quasi-parabolic structure if and only if they differ by a scalar multiple. The above mentioned universal property says that $\mathbb{P}$ is a (fine) moduli space for quasi-parabolic bundles. More precisely, the family of quasi-parabolic structures

$$\mathcal{V} \twoheadrightarrow \mathcal{T}_0$$

parameterized by $\mathbb{P}$ is universal for families of quasi-parabolic bundles

$$\mathcal{E} \twoheadrightarrow \mathcal{T}$$

parameterized by S, whose kernel is a family of semi-stable bundles. The points of $\mathbb{P}$ parameterize quasi-parabolic structures $V \twoheadrightarrow \mathcal{O}_Z$ whose kernel is semi-stable.

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, where $0 < \alpha_1 < \alpha_2 < 1$, and let $\Delta = \Delta_{\boldsymbol{\alpha}}$ be the parabolic datum which attaches weights α_1, α_2 to our quasi-parabolic datum. We can choose α_1 and α_2 so small that

- a parabolic semi-stable bundle is parabolic stable;
- if V is stable, then every parabolic structure on V is parabolic stable;
- the underlying vector bundle of a parabolic stable bundle is semi-stable in the usual sense.

Showing the above involves some very elementary calculations. Call $\boldsymbol{\alpha}$ *small* if α_1 and α_2 satisfy the above properties and denote the resulting (fine) moduli space of parabolic stable bundles $\mathcal{SU}_X(n, L, \Delta)$.

Lemma 5.0.1. *If $\boldsymbol{\alpha}$ is small then for every parabolic stable bundle $V \twoheadrightarrow \mathcal{O}_Z$, the kernel W is semi-stable.*

Proof. Note that $\deg W = -1$ and hence the semi-stability of W is equivalent to its stability. Suppose W is not stable. Then by the above observation, there is a subbundle E of W such that $\mu(E) > \mu(W)$. Let $\operatorname{rk} E = r$. Let $E' \subset V$ be the subbundle generated by E. Let T be the torsion subsheaf of the cokernel of $E \to V$, and t the vector space dimension (over $k(x) = \mathbb{C}$) of T. We then have $\deg E' = \deg E + t$, where $t \geq 0$. Thus

$$-\frac{1}{n} < \mu(E) \leq \mu(E') = \mu(E) + \frac{t}{r} \leq \mu(V) = 0.$$

In particular

$$-\frac{1}{n} < \mu(E) \leq -\frac{t}{r},$$

i.e.

$$\frac{1}{n} > \frac{t}{r},$$

but this is not possible for $t > 0$ since $r < n$. So $t = 0$ and hence $T = 0$. Thus $E = E'$. Let $d = \deg E$. Then

$$-\frac{1}{n} < \frac{d}{r} \leq 0.$$

This is possible only if $d = 0$. Since $E = E'$, one checks that the flag induced on the fibre E'_x by the flag $F_1 V_x \supset F_2 V_x$ is the trivial flag $E'_x = F_1 E'_x = F_2 E'_x$. It follows that the parabolic degree of E' is $r\alpha_2$. Since V is parabolically stable, this means

$$\alpha_2 < \frac{\alpha_1 + (n-1)\alpha_2}{n}.$$

The right side is a non-trivial convex combination of α_1 and α_2, and we also have $\alpha_1 < \alpha_2$. Thus we have a contradiction. Therefore W is stable (see also [3] for the same result when $n = 2$). $\square$

Theorem 5.0.3. *$\mathbb{P} = \mathcal{SU}_X(n, L, \Delta)$, and $\mathcal{V} \twoheadrightarrow \mathcal{T}_0$ is the universal family of parabolic bundles.*

Proof. Let $\mathbb{P}^{ss} \subset \mathbb{P}$ be the locus on which $\mathcal{V}$ consists of parabolic semi-stable (=parabolic stable) bundles. One checks that $\mathbb{P}^{ss}$ is an open subscheme of $\mathbb{P}$ (this involves two things: (i) knowing that the scheme $\widetilde{R}$ of [15, p. 226] has a local universal property for parabolic bundles and (ii) knowing that the scheme $\widetilde{R}^{ss}$ of *loc. cit.* is open).

Clearly $\mathbb{P}^{ss}$ is non-empty — in fact if V is stable of rank n and determinant L, then any parabolic structure on V is parabolic stable (see above). We claim that $\mathbb{P}^{ss} \simeq \mathcal{SU}_X(n,\, L,\, \Delta)$. To that end, let S be a scheme, and

$$\mathcal{E} \twoheadrightarrow \mathcal{T} \tag{5.1}$$

a family of parabolic stable bundles parameterized by S. By Lemma 5.0.1, the kernel $\mathcal{W}'$ of (5.1) is a family of stable bundles of rank n and determinant $L \otimes \mathcal{O}_X(-D)$. Since $\mathcal{S}_1$ is a fine moduli space, we have a unique map $g\colon S \to \mathcal{S}_1$ and a line bundle ξ on S such that $(1 \times g)^*\mathcal{W} = \mathcal{W}' \otimes p_S^*\xi$. By doctoring (5.1) we may assume that $\xi = \mathcal{O}_S$. The universal property of the exact sequence (3.1) on $\mathbb{P}$ then gives us a unique map

$$g\colon\ S \longrightarrow \mathbb{P}$$

such that $(1 \times g)^*(3.1)$ is equivalent to

$$0 \longrightarrow \mathcal{W}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0.$$

Clearly g factors through $\mathbb{P}^{ss}$. This proves that $\mathbb{P}^{ss}$ is $\mathcal{SU}_X(n,\, L,\, \Delta)$. However, $\mathcal{SU}_X(n,\, L,\, \Delta)$ is a projective variety (see [15, pp. 225–226, Theorem 4.1]), whence we have

$$\mathbb{P} = \mathcal{SU}_X(n,\, L,\, \Delta).$$

Clearly, $\mathcal{V} \twoheadrightarrow \mathcal{T}_0$ is the universal family of parabolic bundles. □

Note that the above proof gives $\mathbb{P}^{ss} = \mathbb{P}$, whence we have,

Corollary 5.0.3. *Let*

$$0 \to W \to V \to \mathcal{O}_Z \to 0$$

be an exact sequence of coherent sheaves on X with $W \in \mathcal{S}_1$ and V a vector bundle. Then V is a semi-stable bundle.

It follows that $\mathcal{V}$ consists of (usual) semi-stable bundles (by our choice of $\boldsymbol{\alpha}$). Since $\mathcal{S}$ is a coarse moduli space, we get the map

$$\varphi\colon\ \mathbb{P} \longrightarrow \mathcal{S}. \tag{5.2}$$

Remark 5.0.1. Note that the parabolic structure Δ is something of a red herring. In fact $\mathcal{SU}_X(n,\, L,\, \Delta)$ parameterizes quasi-parabolic structures $V \twoheadrightarrow \mathcal{O}_Z$, whose kernel is semi-stable (cf. [15, p. 238, Remark (5.4)] where this point is made for $n = 2$).

Remark 5.0.2. Let V be a stable bundle of rank n, with $\det V = L$, so that (the isomorphism class of) V lies in $\mathcal{S}^s$. Since any parabolic structure on V is parabolic stable (by our choice of $\boldsymbol{\alpha}$), therefore we see that $f^{-1}(V)$ is canonically isomorphic to $\mathbb{P}(V_x^*)$[2]. Moreover, it is not hard to see that $\mathbb{P}^s := \pi^{-1}(\mathcal{S}^s) \to \mathcal{S}^s$ is smooth (examine the effect on the tangent space of each point on $\mathbb{P}^s$).

[2]One can be more rigorous. Identifying $Z^{\mathbb{P}} = \{x\} \times \mathbb{P}$ with $\mathbb{P}$ we see that restricting the universal exact sequence to $Z^{\mathbb{P}}$ gives us the quotient $\mathcal{O}_{\mathbb{P}} \otimes_{\mathbb{C}} V_x \twoheadrightarrow \mathcal{T}_0|Z^{\mathbb{P}}$. Let S be a scheme which has a quotient $\mathcal{O}_S \otimes_{\mathbb{C}} V_x \twoheadrightarrow \mathcal{L}$ on it where $\mathcal{L}$ is a line bundle. This quotient extends (uniquely) to a family parabolic structures $q_S^*V \twoheadrightarrow \mathcal{T}$ (on V) parameterized by S. By the Lemma, the kernel is a family of stable bundles. The universal property of the exact sequence (3.1) gives us a map $S \to \mathbb{P}$, and this map factors through $f^{-1}(V)$.

5.1. **Codimension estimates.** We wish to estimate $\operatorname{codim}(\mathbb{P}\setminus\mathbb{P}^s)$ as well as $\operatorname{codim}(\mathcal{S}\setminus\mathcal{S}^s)$. The second admits to exact answers (see Remark 5.1.1 below). Heuristically (one can make this rigorous via the deformation theoretic techniques in § 4), the method for obtaining the first estimate is as follows.

Let $V \twoheadrightarrow \mathcal{O}_Z$ be a parabolic bundle in $\mathbb{P}\setminus\mathbb{P}^s$. Then we have a filtration (see [22, p. 18,Théorème 10])

$$0 = V_{p+1} \subset V_p \subset \ldots \subset V_0 = V$$

such that for $0 \le i \le p$, $G_i = V_i/V_{i+1}$ is stable and $\mu(G_i) = \mu$. Moreover (the isomorphism class of) the vector bundle $\bigoplus G_i$ depends only upon V and not on the given filtration. We wish to count the number of moduli at $[V \overset{\theta}{\twoheadrightarrow} \mathcal{O}_Z] \in \mathbb{P}\setminus\mathbb{P}^s$. There are three sources:

a) The choice of $\bigoplus_{i=0}^p G_i$;
b) Extension data;
c) The choice of parabolic structure $V \overset{\theta}{\twoheadrightarrow} \mathcal{O}_Z$, for fixed semi-stable V.

The source c) is the easiest to calculate — there is a codimension one subspace at each parabolic vertex, contributing $n-1$ moduli. Let $n_i = \operatorname{rk} G_i$. The number of moduli arising from a) is evidently

$$\sum_{i=0}^{p}(n_i^2-1)(g-1)+pg.$$

Indeed, the bundles G_i have degree $n_i\mu$ and the product of their determinants must be L. They are otherwise unconstrained. For c), again using techniques in § 4, one sees that the number of moduli contributed by extensions is

$$\begin{aligned}\sum_{i=0}^{p}\left[h^1(G_i^*\otimes V_{i+1})-1\right] &\le \sum_{i=0}^{p}\left[p-i-n_i(n_{i+1}+\ldots n_p)(1-g)\right]-(p+1)\\ &= \frac{p(p+1)}{2}-\sum_{i=0}^{p-1}n_i(n_{i+1}+\ldots+n_p)(1-g)-(p+1)\\ &= \frac{(p+1)(p-2)}{2}-\sum_{i<j}n_in_j(1-g).\end{aligned}$$

This gives

$$\begin{aligned}\operatorname{codim}(\mathbb{P}\setminus\mathbb{P}^s) &\ge \sum_{i<j}n_in_j(g-1)-\frac{(p-1)(p+2)}{2}\\ &= B \qquad \text{(say)}.\end{aligned}$$

Now, $\sum_{i<j} n_in_j \ge p(p+1)/2$, therefore

$$B \ge \frac{p(p+1)}{2}(g-1)-\frac{(p+2)(p-1)}{2}.$$

It follows that $B \ge 3$ whenever $p \ge 2$ *and* $g \ge 3$. If $p=1$ and $n \ge 3$, then

$$B/(g-1) = \sum_{i<j} n_in_j \ge 2$$

and one checks that $B \ge 3$ whenever $g \ge 3$.

Remark 5.1.1. We could use similar techniques to estimate $\operatorname{codim}(\mathcal{S} \setminus \mathcal{S}^s)$, but our task is made easier by the exact answers in [22, p. 48, A]. For just this remark, assume $d > n(2g-1)$, and let $a = (n, d)$. Then $a \geq 2$. Let $n_0 = n/a$. Then according to *loc. cit.* ,

$$\operatorname{codim}(\mathcal{S} \setminus \mathcal{S}^s) = \begin{cases} (n^2-1)(g-1) - \dfrac{n^2}{2}(g-1) - 2 + g & \text{if } a \text{ is even} \\ \\ (n^2-1)(g-1) - \dfrac{n^2+n_0^2}{2}(g-1) - 2 + g & \text{if } a \text{ is odd.} \end{cases}$$

It now follows that

$$\operatorname{codim}(\mathcal{S} \setminus \mathcal{S}^s) > 5$$

if n, g are in the range of Theorem 1.0.2(b).

6. Hodge theory

This section, which can be read independently of the rest of the paper, contains some results from Hodge theory that will be needed to complete the proofs of the main theorems.

6.1. **Purity.** We refer to [8] for the definition and basic properties of mixed Hodge structures. Deligne's fundamental result is that the cohomology groups of schemes with coefficients in $\mathbb{Z}$ carry canonical mixed Hodge structures. We will need certain purity results for these mixed Hodge structure for low degree cohomology of smooth open varieties. These results can be deduced by comparing ordinary cohomology to intersection cohomology and appealing to the work Saito [20]. However we will give more elementary arguments, using a version of the Lefschetz hyperplane theorem.

The notation of this section will be independent of the others.

Lemma 6.1.1. *If Y is a smooth variety, Z a codimension k closed subscheme, and $U = Y \setminus Z$, then*

$$H^j(Y, \mathbb{Z}) \xrightarrow{\sim} H^j(U, \mathbb{Z})$$

for $j < 2k-1$.

Proof. We have to show that $H^j_Z(Y, \mathbb{Z})$ vanishes for $j < 2k$. By Alexander duality (see for e.g. [12, p. 381, Theorem 4.7]) we have

$$H^j_Z(Y, \mathbb{Z}) \xrightarrow{\sim} H_{2m-j}(Z, \mathbb{Z}),$$

where $m = \dim Y$ and H_* is Borel-Moore homology. Now use [12, p. 406, 3.1] to conclude that the right side vanishes if $j < 2k$ (note that "dim" in *loc.cit* is dimension as an analytic space, and in *op.cit.* it is dimension as a topological (real) manifold). □

Remark 6.1.1. In view of the above Lemma, it seems that Balaji's proof of Torelli (for Seshadri's desingularization of $\mathcal{SU}_X(2, \mathcal{O}_X)$) does not work for $g = 3$, for in this case, the codimension of $\mathbb{P} \setminus \mathbb{P}^s = 2$. (See [4, top of p. 624] and [3, Remark 9].)

Corollary 6.1.1. *$H^j(U, \mathbb{Z})$ is pure of weight j when $j < 2k-1$.*

We will need purity results even when compactification is not smooth. To this end we prove the following version of the Lefschetz theorem.

Theorem 6.1.1. *Let Y be an m-dimensional projective variety. Suppose that U is a smooth Zariski open subset. If H is a hyperplane section of Y such that $U \cap H$ is non-empty, then*

$$H^i(U, \mathbb{Q}) \to H^i(U \cap H, \mathbb{Q})$$

is an isomorphism for $i < m-1$ and injective when $i = m-1$.

Proof. We need some results involving Verdier duality. The standard references are [6] and [12]. Let S be an analytic space and p_S the map from S to a point. For $\mathcal{F} \in D^b_{const}(S, \mathbb{Q})$ (the derived category of bounded complexes of $\mathbb{Q}_S$-sheaves whose cohomology sheaves are $\mathbb{Q}_S$-constructible), set

$$D_S(\mathcal{F}) = \mathbb{R}\mathcal{H}\mathrm{om}_S(\mathcal{F}, p_S^!\mathbb{Q}).$$

We then have by Verdier duality

$$\mathbb{H}^i(S, \mathcal{F}) \xrightarrow{\sim} \mathbb{H}^{-i}(S, D_S(\mathcal{F}))^*. \tag{6.1}$$

Here $\mathbb{H}^*$ denotes "hypercohomology".

For an open immersion $h\colon S' \hookrightarrow S$, one has canonical isomorphisms

$$\mathbb{R}h_* D_{S'}\mathcal{G} \xrightarrow{\sim} D_S(h_!\mathcal{G}) \tag{6.2}$$

and

$$\mathbb{R}h_! D_{S'}\mathcal{G} \xrightarrow{\sim} D_S(\mathbb{R}h_*\mathcal{G}) \tag{6.3}$$

Here $\mathcal{G} \in D^b_{const}(S', \mathbb{Q})$. The first isomorphism is easy (using Verdier duality for the map h) and the second follows from the first and from the fact that $D_{S'}$ is an involution. We have used (throughout) the fact that $h_!$ is an exact functor.

If S is smooth, we have

$$p_S^!\mathbb{Q} = \mathbb{Q}_S[2\dim S]. \tag{6.4}$$

In order to prove the theorem, let $V = U \setminus H$ and $W = Y \setminus H$. We then have a cartesian square

$$\begin{array}{ccc} V & \xrightarrow{\imath'} & U \\ {\scriptstyle \jmath'}\downarrow & & \downarrow{\scriptstyle \jmath} \\ W & \xrightarrow[\imath]{} & Y \end{array}$$

where each arrow is the obvious open immersion. We have, by (6.2) and (6.3), the identity

$$\jmath_!\mathbb{R}\imath'_* D_V\mathbb{Q}_V = D_Y(\mathbb{R}\jmath_*\imath'_!\mathbb{Q}_V). \tag{6.5}$$

Consider the exact sequence of sheaves

$$0 \longrightarrow \imath'_!\mathbb{Q}_V \longrightarrow \mathbb{Q}_U \longrightarrow g_*\mathbb{Q}_{H\cap U} \longrightarrow 0$$

where $g\colon H \cap U \to U$ is the natural closed immersion. It suffices to prove that $H^i(U, \imath'_!\mathbb{Q}_V) = 0$ for $i \le m-1$. Now,

$$H^i(U, \imath'_!\mathbb{Q}_V) = \mathbb{H}^i(Y, \mathbb{R}\jmath_*\imath'_!\mathbb{Q}_V).$$

Using (6.1), (6.5) and (6.4), the above is dual to

$$\mathbb{H}^{-i}(Y, \jmath_!\mathbb{R}\imath'_* D_V\mathbb{Q}_V) = \mathbb{H}^{2m-i}(Y, \jmath_!\mathbb{R}\imath'_*\mathbb{Q}_V)$$

But $\jmath_! \mathbb{R}\imath'_* = \mathbb{R}\imath_* \jmath'_!$, and hence the above is

$$\begin{aligned} &= \mathbb{H}^{2m-i}(Y,\, \mathbb{R}\imath_*(\jmath'_!\mathbb{Q}_V)) \\ &= \mathbb{H}^{2m-i}(W,\, \jmath'_!\mathbb{Q}_V) \\ &= H^{2m-i}(W,\, \jmath'_!\mathbb{Q}_V). \end{aligned}$$

Now, W is an affine variety, and therefore, according to M. Artin, its constructible cohomological dimension is less than or equal to its dimension [1]. Consequently, the above chain of equalities vanish whenever $i < m$ (see also [10]). ☐

We will use the notation of this theorem for the remainder of the section. We immediately have:

Corollary 6.1.2. *Let $e = \operatorname{codim}(Y \setminus U)$. For $i < e - 1$, the Hodge structure $H^i(U,\, \mathbb{Z})$ is pure of weight i.*

Proof. This is true if U is projective. In general proceed using Bertini's theorem, induction, Theorem 6.1.1 and the fact that submixed Hodge structures of pure Hodge structures are pure [8]. ☐

6.2. **Polarizations.** Let Y, U, e, etc. be as in Theorem 6.1.1. Let $i \in \mathbb{N}$ and $\mathcal{L}$ a line bundle on Y be such that

(a) $H^j(U,\, \mathbb{Q}) = 0$ for $j = i-2,\, i-4, \ldots$ (note that this forces i to be *odd*);
(b) $i < e - 1$;
(c) $\mathcal{L}$ is very ample.

Let M be the intersection of $k = m - e + 1$ hyperplanes in general position. Then M is a smooth variety contained in U. Let

$$l \colon H^i(U) \longrightarrow H_c^{2m-i}(U)$$

be the composite of

$$\begin{aligned} H^i(U) &\longrightarrow H^i(M) \\ &\longrightarrow H^{2m-2k-i}(M) \\ &\longrightarrow H_c^{2m-i}(U) \end{aligned}$$

where the first map is restriction, the second is "cupping with $c_1(\mathcal{L})^{m-k-i}$" and the third is the Poincaré dual to restriction. The map l is also described as

$$x \mapsto x \cup c_1(\mathcal{L})^{m-k-i} \cup [M].$$

One then has (easily)

Lemma 6.2.1. *If M' is another k-fold intersection of general hyperplanes, then $[M'] = [M]$. Therefore l depends only on $\mathcal{L}$.*

Proposition 6.2.1. *The pairing*

$$< x,\, y >= \int_U l(x) \cup y$$

on $H^i(U,\, \mathbb{C})$ gives a polarization on the pure Hodge structure $H^i(U,\, \mathbb{Z})$. This makes the associated complex torus $J^p(U)$ (where $i = 2p-1$) into an abelian variety when $H^i(U)$ is of type $\{(p, p-1), (p-1, p)\}$.

Proof. By Theorem 6.1.1, we have an isomorphism

$$r\colon H^i(U) \longrightarrow H^i(M).$$

The latter Hodge structure carries a polarization given by

$$< \alpha,\, \beta >= \int_M c_1(\mathcal{L})^{m-k-i} \cup \alpha \cup \beta.$$

The conditions on i and the Hodge-Riemann bilinear relations on the primitive part of $H^i(M,\, \mathbb{C})$, assure us that the above is indeed a polarization (see [11] or [25, Chap. V, §6]). Our conditions on i imply that the primitive part of $H^i(M)$ is everything so that the p-th intermediate Jacobian of M is an abelian variety. The pairing on $H^i(M)$ translates to a polarization on $H^i(U)$ given by

$$< x,\, y >= \int_U l(x) \cup y.$$

This gives the result. □

6.3. **Hodge structure of projective bundles.** Deligne's construction [8] gives a stronger result than is actually stated, namely that for a smooth quasi-projective variety X, $H^i(X)$ take values in the subcategory of polarizable mixed Hodge structures [5]. One pleasant feature of this subcategory is the following generalization of Poincaré reducibility:

Lemma 6.3.1. *The category of polarizable rational pure Hodge structures is semisimple.*

Proof. This follows from [5]. □

Corollary 6.3.1. *The category of polarizable rational pure Hodge structures satisfies cancellation, i.e. if $A \oplus B \cong A \oplus C$ then $B \cong C$.*

Proposition 6.3.1. *Let $p : M \to N$ be a $(d-1)$-fold fiber product of $\mathbb{P}^{n-1}$-bundles (which need not be locally trivial in the Zariski topology) over a smooth simply connected quasiprojective variety N. Then*

(a) $H^1(M, \mathbb{Z}) = 0$*;*
(b) $p^* : H^3(N, \mathbb{Z}) \to H^3(M, \mathbb{Z})$ *is surjective, and its kernel is finite. In particular p^* induces an isomorphism $H^3(N,\, \mathbb{Z})_{free} \cong H^3(M,\, \mathbb{Z})_{free}$ (where $A_{free} = A/A_{tors}$)*
(c) $H^i(M, \mathbb{Q}) = \bigoplus_{j\geq 0} H^{i-2j}(N, \mathbb{Q}) \otimes H^{2j}((\mathbb{P}^{n-1})^{d-1}, \mathbb{Q})$ *as mixed Hodge structures.*

Proof. We will use the Leray spectral sequence with coefficients in $\mathbb{Z}$ and $\mathbb{Q}$. As N is simply connected, $R^i p_* \mathbb{Z}$ is the constant sheaf $H^i((\mathbb{P}^{n-1})^{d-1}, \mathbb{Z})_N$. The first statement is an immediate consequence of the Leray spectral sequence since $H^1(N,\, p_*\mathbb{Z}) = 0$.

Note that aside from $H^3(N,\, p_*\mathbb{Z}) = H^3(N,\, \mathbb{Z})$, all the other E_2 terms that contribute to $H^3(M, \mathbb{Z})$ vanish. This implies that $H^3(M,\, \mathbb{Z})$ is a quotient of $H^3(N,\, \mathbb{Z})$. But it is an isomorphism after tensoring with $\mathbb{Q}$ by [7]. Since the quotient $H^3(N,\, \mathbb{Z}) \twoheadrightarrow H^3(M,\, \mathbb{Z})$ arising from the analysis of the spectral sequence is precisely p^*, therefore the kernel is finite. This implies that the induced map $H^3(N,\, \mathbb{Z})_{free} \to H^3(M,\, \mathbb{Z})_{free}$ is surjective with trivial kernel.

Suppose that $p : M \to N$ is a $\mathbb{P}^{n-1}$ bundle, that is $d = 2$. Let D be a "nonvertical" divisor class on M i.e. a class which restricts nontrivially to each fiber of p (for

instance $c_1(\omega_{M/N})$). Then the subspace generated by D^i can be identified with a copy of $\mathbb{Q}(-i)$ in $H^{2i}(M)$. Thus one gets a morphism of mixed Hodge structures,

$$\bigoplus_{j<n} H^{i-2j}(N)(-j) \to H^i(M)$$

given by summing the maps $\alpha \mapsto p^*\alpha \cup D^j$. Note that the restrictions of D^j generate the cohomology of each fiber. Therefore by the Leray-Hirch theorem the above maps are isomorphisms, and this proves the last statement in this case. In general, M is a fiber product of $\mathbb{P}^{n-1}$ bundles. Let D_k be the pullback of a nonvertical divisor class from the kth factor. Arguing as before, we get isomorphisms

$$\bigoplus_{j_1+\ldots j_{d-1}=j,\, j_i<n} H^{i-2j}(N)(-j) \to H^i(M)$$

obtained by summing the maps

$$\alpha \mapsto p^*\alpha \cup D_1^{j_1} \cup \ldots D_{d-1}^{j_{d-1}}$$

□

By combining this with the previous corollary, and using the fact that sub-mixed Hodge structures of pure Hodge structures are pure, we obtain:

Corollary 6.3.2. *Let $p_i : M_i \to N_i$ $i = 1, 2$ be two $(\mathbb{P}^{n-1})^{d-1}$-bundles satisfying the hypotheses on p in Proposition 6.3.1. If $\phi : M_1 \to M_2$ is a morphism of varieties inducing isomorphisms (of rational mixed Hodge structures) $H^i(M_2, \mathbb{Q}) \to H^i(M_1, \mathbb{Q})$ for $i \le k$, and if these mixed Hodge structures are pure of weight i then for $i \le k$ the Hodge structures $H^i(N_2, \mathbb{Q})$ and $H^i(N_1, \mathbb{Q})$ are pure of weight i and are* non-canonically *isomorphic as rational pure Hodge structures of weight i.*

7. Hodge structure on degree one moduli

Let $\mathcal{U}_X(n, 1)$ be the moduli space of semi-stable rank n vector bundles of degree 1 on X. There is a smooth morphism $\det : \mathcal{U}_X(n, 1) \to Pic^1(X)$ given by the determinant, and the fiber over L' is just $\mathcal{SU}_X(n, L')$.

Atiyah and Bott [2, 9.11] gave generators for the cohomology ring $H^*(\mathcal{U}_X(n, 1), \mathbb{Q})$ in terms of certain universal characteristic classes. As the restriction map on rational cohomology from $\mathcal{U}_X(n, 1)$ to $\mathcal{SU}_X(n, L')$ is surjective [loc. cit. 9.7], these give generators for the cohomology of the $\mathcal{SU}_X(n, L')$. We will describe these generators in a form which is convenient for us. Let E be a Poincaré vector bundle. Fix a line bundle $L' \in Pic^1(X)$, let p_i denote the projections on $X \times \mathcal{SU}_X(n, L')$ and let c_r be the r-th Chern class of E restricted to $X \times \mathcal{SU}(n, L')$. Let $\gamma_{r,i}$ denote the composition of the following morphisms of Hodge structures:

$$\begin{aligned} H^i(X, \mathbb{Q})(-r+1) &\xrightarrow{p_1^*} H^i(X \times \mathcal{SU}_X(n, L'), \mathbb{Q})(-r+1) \\ H^i(X \times \mathcal{SU}_X(n, L'), \mathbb{Q})(-r+1) &\xrightarrow{c_r \cup} H^{i+2r}(X \times \mathcal{SU}_X(n, L'), \mathbb{Q})(1) \\ H^{i+2r}(X \times \mathcal{SU}_X(n, L'), \mathbb{Q})(1) &\xrightarrow{p_{2*}} H^{i+2r-2}(\mathcal{SU}_X(n, L'), \mathbb{Q}) \end{aligned}$$

Theorem 7.0.1. (Atiyah-Bott [2]) *The ring $H^*(\mathcal{SU}_X(n, L'), \mathbb{Q})$ is generated by the images of the maps $\gamma_{r,i}$ for $2 \le r \le n$ and $0 \le i \le 2$.*

Corollary 7.0.3. *For each i, any simple summand of the Hodge structure*

$$H^i(\mathcal{SU}_X(n,L'),\mathbb{Q})$$

is, up to Tate twisting, a direct summand of a tensor power of $H^1(X)$. The Hodge structure on $H^i(\mathcal{SU}_X(n,L'),\mathbb{Q})$ is independent of $L'\in Pic^1(X)$

Proof. Let $P = Pic^1(X)$. Let H be the direct sum of all the tensor products

$$H^{i_1}(X,\mathbb{Q})(-r_1+1)\otimes H^{i_2}(X,\mathbb{Q})(-r_2+1)\ldots$$

indexed by the finite sequences $((i_1,r_1),(i_2,r_2),\ldots)$ with

$$(i_1+2r_1-2)+(i_2+2r_2-2)+\ldots = i.$$

Then, by the theorem, there is a surjection $H\to H^i(\mathcal{SU}_X(n,L'),\mathbb{Q})$ given by the product of γ's. This implies the first statement. The above map extends to a surjection of variations of Hodge structures $H_P\to R^i\det_*\mathbb{Q}$, where the H_P denotes the constant variation with fiber H. The second statement now follows from the theorem of the fixed part [8, 4.1.2].

□

Since the relations among the above generators have recently been determined by Jeffrey and Kirwan [13], it is possible to make a complete determination of these Hodge structures (over $\mathbb{Q}$). It is not clear whether the maps $\gamma_{r,i}$ are surjective for integral cohomology, however one does have:

Theorem 7.0.2. (Narasimhan-Ramanan [18]) *The map*

$$\gamma_{2,1}: H^1(X,\mathbb{Z})(-1)\to H^3(\mathcal{SU}_X(n,L'),\mathbb{Z})$$

is an isomorphism.

This is not stated as such, but this is implicit in their proof of their third theorem.

8. Main theorems

8.1. **Natural polarizations.** We give a proof of the following "folklore" theorem: The theta divisor and its multiples are the only natural polarizations on the Jacobian. To simplify the discussion, we work with polarizations in the Hodge theoretic sense. Let $\pi:\mathcal{X}\to T$ be a family of genus g curves over an irreducible base variety. Let θ be the standard polarization on $R^1\pi_*\mathbb{Z}$ corresponding to cup product, and let θ' be some other polarization.

Lemma 8.1.1. *If the canonical map to moduli space $T\to M_g$ is dominant, then there exist a positive integer m such that $\theta' = m\theta$.*

Proof. Let $t_0\in T$ be a base point. θ_{t_0} can be viewed as a primitive vector in $V = \wedge^2 H^1(X_{t_0},\mathbb{Z})^{\pi_1(T,t_0)}$. Therefore it is enough to prove that $V\otimes\mathbb{R}$ is one dimensional. After replacing T by a Zariski open subset, we can assume that the image $T'\subset M_g$ is disjoint from the locus of curves with automorphisms, and that $T\to T'$ is smooth. This guarantees (by Teichmuller theory) that $\pi_1(T')$ surjects onto the mapping class group which surjects onto $Sp_{2g}(\mathbb{Z})$, and furthermore that the image of $\pi_1(T)$ has finite index in $\pi_1(T')$. Therefore $\pi_1(T)$ has Zariski dense image in $Sp_{2g}(\mathbb{R})$. Consequently $V\otimes\mathbb{R} = \wedge^2 H^1(X_{t_0},\mathbb{R})^{Sp_{2g}(\mathbb{R})}$ which is well known to be spanned by the standard symplectic form. □

8.2. **First main theorem.**

Theorem 8.2.1. *Let* $\imath(n,g) = 2(n-1)g-(n-1)(n^2+3n+1)-7$*. Let* X *be a curve of genus* $g \geq 2$*. If* $n \geq 4$ *and* $i < \imath(n,g)$ *are integers, then for any pair of line bundles* L, L' *on* X*,the mixed Hodge structures* $H^i(\mathcal{SU}_X^s(n,L),\mathbb{Q})$ *and* $H^i(\mathcal{SU}_X^s(n,L'),\mathbb{Q})$ *are* (noncanonically) *isomorphic and are both pure of weight* i*.*

Proof. There is no loss of generality in assuming that deg $L' = 1$. In fact, by 7.0.3, we can assume that L' is a specific line bundle, namely $L' = L(-D)$, where $D = x^1 + \dots x^{d-1}$. Then reverting to the our earlier notation, we have $\mathcal{S}_1 = \mathcal{SU}_X(n,L')$ and $\mathcal{S} = \mathcal{SU}_X(n,L)$. Consider the diagram

$$\mathcal{S}_1 \xleftarrow{\pi} \mathbb{P} \supset U \xrightarrow{f} U' \subset \mathcal{S}^s.$$

Then corollaries 4.4.1, 4.4.2 and Lemma 6.1.1 implies

$$H^i(\mathbb{P},\mathbb{Q}) \cong H^i(U,\mathbb{Q})$$

$$H^i(U',\mathbb{Q}) \cong H^i(\mathcal{S}^s,\mathbb{Q})$$

for all $i < \imath(n,g)$. The theorem follows from Corollary 6.3.2 (see also Remark 3.3.1). □

8.3. **Second main theorem.**

Theorem 8.3.1. *Let* X *be a curve of genus* $g \geq 2$*,* $n \geq 2$ *an integer and* L *a line bundle on* X*. Let* $\mathcal{S}^s = \mathcal{SU}_X^s(n,\, L)$*.*

(a) *If* $g > \frac{3}{n-1} + \frac{n^2+3n+3}{2}$ *and* $n \geq 4$*, then* $H^3(\mathcal{S}^s,\mathbb{Z})$ *is a pure Hodge structure of type* $\{(1,2),\,(2,1)\}$*, and it carries a natural polarization making the intermediate Jacobian*

$$J^2(\mathcal{S}^s) = \frac{H^3(\mathcal{S}^s,\,\mathbb{C})}{F^2 + H^3(\mathcal{S}^s,\,\mathbb{Z})}$$

into a principally polarized abelian variety. There is an isomorphism of principally polarized abelian varieties $J(X) \simeq J^2(\mathcal{S}^s)$*.*

(b) *If* deg L *is a multiple of* n*, then the conclusions of* (a) *are true for* $g \geq 3$*,* $n \geq 2$ *except the case* $g = 3, n = 2$*.*

Proof. We concentrate on part (a). The proof of part (b) is identical if we take the Hecke correspondence of § 5, and the codimension estimates of $\mathbb{P} \setminus \mathbb{P}^s$ and $\mathcal{S} \setminus \mathcal{S}^s$ given in that section (see 5.1). Consider the diagram

$$\mathcal{S}_1 \xleftarrow{\pi} \mathbb{P} \supset U \xrightarrow{f} U' \subset \mathcal{S}^s.$$

once again. Then corollaries 4.4.1, 4.4.2 and Lemma 6.1.1 imply

$$H^3(\mathbb{P},\mathbb{Z}) \cong H^3(U,\,\mathbb{Z}),$$

$$H^3(U',\mathbb{Z}) \cong H^3(\mathcal{S}^s,\,\mathbb{Z}).$$

$\mathcal{S}_1$ is unirational (see [22, pp. 52–53, VI.B]), hence so is $\mathbb{P}$. Therefore these varieties are simply connected [21]. Since codim$(\mathbb{P} \setminus U) > 1$ it follows that U is simply connected (purity of branch locus). The homotopy exact sequence tells us that U' is simply connected. Proposition 6.3.1 applied to π and f implies that on the third cohomology, π^* and f^* are surjective with finite kernels. Since π is locally trivial

in the Zariski topology, π^* is in fact an isomorphism. Combining these facts with the isomorphisms above produces a map of Hodge structures

$$H^3(\mathcal{S}^s,\, \mathbb{Z}) \to H^3(\mathcal{S}_1,\, \mathbb{Z})$$

which is surjective with finite kernel. As an immediate consequence we have,

$$H^3(\mathcal{S}_1, \mathbb{Z})_{free} \cong H^3(\mathcal{S}^s, \mathbb{Z})_{free}.$$

This, together with with Theorem 7.0.2, yields an isomorphism of Hodge structures:

$$H^1(X, \mathbb{Z})(-1) \cong H^3(\mathcal{S}^s, \mathbb{Z})_{free} \tag{8.1}$$

which yields an isomorphism of tori $J(X) \cong J^2(\mathcal{S}^s)$.

The next step is to construct a polarization on $H^3(\mathcal{S}^s, \mathbb{Z})$. One knows from the results of Drezet and Narasimhan [9], that $\mathrm{Pic}(\mathcal{S}^s) = \mathbb{Z}$ (see p. 89, 7.12 (especially the proof) of *loc.cit.*). Moreover, $\mathrm{Pic}(\mathcal{S}) \to \mathrm{Pic}(\mathcal{S}^s)$ is an isomorphism. Let ξ' be the ample generator of $\mathrm{Pic}(\mathcal{S}^s)$. It is easy to see that there exists a positive integer r, independent of $(X,\, L)$ (with genus $X = g$), such that $\xi = {\xi'}^r$ is very ample on $\mathcal{S}$ (we are not distinguishing between line bundles on $\mathcal{S}^s$ and their (unique) extensions to $\mathcal{S}$). Embed $\mathcal{S}$ in a suitable projective space via ξ. Let $e = \mathrm{codim}(\mathcal{S} \setminus \mathcal{S}^s)$. Let M be the intersection of $k = \dim \mathcal{S} - e + 1$ hyperplanes (in general position) with $\mathcal{S}^s$. Then M is smooth, projective and contained in $\mathcal{S}^s$. Let $p = \dim \mathcal{S}$ and H^*_c — cohomology with compact support. We then have a map

$$l\colon H^3(\mathcal{S}^s) \longrightarrow H^{2p-3}_c(\mathcal{S}^s)$$

defined by

$$x \mapsto x \cup c_1(\xi)^{p-k-3} \cup [M].$$

If M' is another k-fold intersection of general hyperplanes, then $[M'] = [M]$. Hence l depends only on ξ. According to Proposition 6.2.1, the pairing on $H^3(\mathcal{S}^s,\, \mathbb{C})$ given by

$$< x,\, y >= \int_{\mathcal{S}^s} l(x) \cup y$$

gives a polarization on the Hodge structure of $H^3(\mathcal{S}^s)$. Pulling this back via the isomorphism (8.1) gives a second polarization on $H^1(X)$. If we can show that $<,>$ varies over the whole M_g, then we can appeal to lemma 8.1.1 to show that there exists a positive integer m (independent of X) such that $\frac{1}{m} <,>$ coincides with the standard principal polarization of $H^1(X)$, and this will complete the proof.

Let T be the moduli space parameterizing tuples

$$(X, x^1, \ldots x^{d-1}, L, \lambda)$$

consisting of a genus g curve, $d-1$ distinct points, a degree d line bundle and a level 3 structure. T is an irreducible variety which surjects onto M_g. The inclusion of the level structure guarantees that T is fine, and therefore there is a universal curve $\mathcal{X} \to T$ together with $d-1$ sections and so on. All of the constructions given so far can be carried out relative to T, in other words we can construct a diagram of T-schemes

$$\begin{array}{ccccc} S_1 & \leftarrow & P & \dashrightarrow & S \\ & \searrow p_1 & \downarrow & p \swarrow & \\ & & T & & \end{array}$$

whose fibers are the Hecke correspondences. Furthermore there is polarization on $R^3p_*\mathbb{Z} \cong R^1p_{1*}\mathbb{Z}(-1)$, which restricts to the one constructed in the previous paragraph. We now in a position to apply lemma 8.1.1 to conclude the proof.

□

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